2.14 Series

Summation Notation

In **summation notation,** the sum of the terms of the sequence $\{a_1, a_2, a_3, \ldots a_n\}$ is denoted:

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \ldots + a_n$$

Which is read "the sum of a_k from k = 1 to n."

The symbol Σ is simply an instruction to sum, or add up, the terms. The integer k is called the index of the sum, it tells you where to start the sum and where to end it. The expression

$\displaystyle\sum_{k=1}^{n} a_k$ is the instruction to add the terms a_k of the sequence $\{a_n\}$ from k = 1 through k = n.

Add all the numbers from 1 to 100.

(Story of Gauss)

1 + 2 + 3 + 4 + 5 + … + 99 + 100

100 + 99 + 98 + … + 2 + 1

101 X 100 = 10,100

10,100/2 = 5,050

Sum of n terms of an Arithmetic Sequence

Let $\{a_n\}$ be an arithmetic sequence with first term a_1 and common difference d. The sum S_n of the first n terms of $\{a_n\}$ is:

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \ldots + a_n$$

$$= n\left(\frac{a_1 + a_n}{2}\right)$$

$$= \frac{n}{2}\left(2a_1 + (n-1)d\right)$$

Example:

A corner section of a stadium has 8 seats along the front row. Each successive row has two more seats than the row preceding it. If the top row has 24 seats, how many seats are in the entire section?

The numbers of seats in the rows form an arithmetic sequence with

$$a_1 = 8,\ a_n = 24,\ \text{and}\ \ d = 2.$$

Solving $a_n = a_1 + (n-1)d$, we find that

$$\begin{aligned} 24 &= 8 + (n-1)(2) \\ 16 &= (n-1)(2) \\ 8 &= n-1 \\ n &= 9 \end{aligned}$$

Applying the Sum of a Finite Arithmetic Sequence Theorem, the total number of seats in the section is 9(8 + 24)/2 = 144.

Sum of a Finite Geometric Sequence:

Let $\{a_n\}$ be a finite geometric sequence with common ratio r $\neq$ 1. Then the sum of the terms of the sequence is:

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \ldots + a_n$$

$$= \frac{a_1(1-r^n)}{1-r}$$

Example:

Find the sum of the geometric sequence 4, -4/3, 4/9, -4/27, . . . , $4(-1/3)^{10}$.

We can see that $a_1 = 4$ and $r = -1/3$. The *n*th term is $4(-1/3)^{10}$, which means that $n = 11$. (Remember that the exponent on the *n*th term is $n - 1$, not n.) Apply the Sum of a Finite Geometric Sequence Theorem, we find that

$$\sum_{n=1}^{11} 4\left(\frac{-1}{3}\right)^{n-1} = \frac{4\left(1-\left(\frac{-1}{3}\right)^{11}\right)}{1-\left(\frac{-1}{3}\right)} \approx 3.000016935.$$

Infinite Series:

An infinite series is an expression of the form

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \ldots + a_n + \ldots$$

In some cases the sequence of **partial sums** approaches a finite limit S:

$$\lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} (a_1 + a_2 + \ldots a_n) = S$$

In this case we say that the series **converges** to S, and it makes sense to define S as the **sum of the infinite series**.

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = S$$

If the limit of a partial sum does not exist, then the series **diverges** and has no sum.

Example:

For each of the following series, find the first five terms in the sequence of partial sums. Which of the series appear to converge?

a) 0.1 + 0.01 + 0.001 + 0.0001 + . . .

b) 10 + 20 + 30 + 40 + . . .

c) 1 – 1 + 1 – 1 + . . .

a) The first five partial sums are {0.1, 0.11, 0.111, 0.1111, 0.11111}. These appear to be approaching a limit of $0.\bar{1} = \frac{1}{9}$, which would suggest that the series converges to a sum of 1/9.

b) The first five partial sums are {10, 30, 60, 100, 150}. These numbers increase without bound and do not approach a limit. The series diverges and has not sum.

c) The first five partial sums are {1, 0, 1, 0, 1}. These numbers oscillate and do not approach a limit. The series diverges and has no sum.

Sum of an Infinite Geometric Series:

The geometric series $\sum_{k=1}^{\infty} a \cdot r^{k-1}$ converges if and only if $|r| < 1$. If it does converge, the sum is

$$S = \frac{a}{1-r}.$$

Example:

Determine whether the series converges. If it converges, give the sum.

$a)\ \sum_{k=1}^{\infty} 3(0.75)^{k-1}$ $\qquad$ $b)\ \sum_{n=0}^{\infty}\left(-\frac{4}{5}\right)^n$

$c)\ \sum_{n=1}^{\infty}\left(\frac{\pi}{2}\right)^n$ $\qquad$ $d)\ 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$

a) Since $|r| = |0.75| < 1$, the series converges. The first term is $3(0.75)^0 = 3$, so the sum is $a/(1-r) = 3(1-0.75) = 12$.

b) Since $|r| = |-4/5| < 1$, the series convergers. The firs term is $(-4/5)^0 = 1$, so the sum is $a/(1-r) = 1/(1-(-4/5)) = 5/9$.

c) Since $|r| = \left|\frac{\pi}{2}\right| > 1$, the series diverges.

d) Since $|r| = \left|\frac{1}{2}\right| < 1$, the series converges. The first term is 1, and so the sum is $a/(1-r) = 1/(1-1/2) = 2$.